

EQUIVARIANT CLASSIFICATION OF 2-TORUS MANIFOLDS

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1. INTRODUCTION

In this paper, we consider the equivariant classification of locally standard 2-torus manifolds. A 2-torus manifold is a closed smooth manifold of dimension n with an effective action of a 2-torus group $(\mathbb{Z}_2)^n$ of rank n , and it is said to be locally standard if it is locally isomorphic to a faithful representation of $(\mathbb{Z}_2)^n$ on \mathbb{R}^n . The orbit space Q of a locally standard 2-torus M by the action is a nice manifold with corners. When Q is a simple convex polytope, M is called a small cover and studied in [4]. A typical example of a small cover is a real projective space $\mathbb{R}P^n$ with a standard action of $(\mathbb{Z}_2)^n$. Its orbit space is an n -simplex. On the other hand, a typical example of a compact non-singular toric variety is a complex projective space $\mathbb{C}P^n$ with a standard action of $(\mathbb{C}^*)^n$ where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. $\mathbb{C}P^n$ has complex conjugation and its fixed point set is $\mathbb{R}P^n$. More generally, any compact non-singular toric variety admits complex conjugation and its fixed point set often provides an example of a small cover. Similarly to the theory of toric varieties, an interesting connection among topology, geometry and combinatorics is discussed for small covers in [4], [5] and [7]. Although locally standard 2-torus manifolds form a much wider class than small covers, one can still expect such a connection. See [9] for the study of 2-torus manifolds from the viewpoint of cobordism.

The orbit space Q of a locally standard 2-torus manifold M contains a lot of topological information on M . For instance, when Q is a simple convex polytope (in other words, when M is a small cover), the betti numbers of M (with \mathbb{Z}_2 coefficient) are described in terms of face numbers of Q ([4]). This is not the case for a general Q , but the euler characteristic of M can be described in terms of Q (Theorem 4.1). Although Q contains a lot of topological information on M , Q is not sufficient to reproduce M , i.e., there are many locally standard 2-torus manifolds with the same orbit space in general. We need two data to reproduce M from Q . One is a characteristic function on Q introduced in [4]. It is a map from the set of codimension-one faces of Q to $(\mathbb{Z}_2)^n$ satisfying a certain linearly independence condition. Roughly speaking, a characteristic function provides information on the set of non-free orbits in M . The other data is a principal $(\mathbb{Z}_2)^n$ -bundle over Q which provides information on the set of free orbits in M . It turns out that the orbit space Q together with these two data uniquely determines a locally standard 2-torus manifold up to equivariant homeomorphism (Lemma 3.1).

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When Q is a simple convex polytope, any principal $(\mathbb{Z}_2)^n$ -bundle over it is trivial; so only a characteristic function matters in this case ([4]).

The set of isomorphism classes in all principal $(\mathbb{Z}_2)^n$ -bundles over Q can be identified with $H^1(Q; (\mathbb{Z}_2)^n)$. Let $\Lambda(Q)$ be the set of all characteristic functions on Q . Then each element in $H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)$ determines a locally standard 2-torus manifold with orbit space Q . However, different elements in the product may produce equivariantly homeomorphic locally standard 2-torus manifolds. Let $\text{Aut}(Q)$ be the group of self-homeomorphisms of Q as a manifold with corners. It naturally acts on $H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)$ and one can see that equivariant homeomorphism classes in locally standard 2-torus manifolds with orbit space Q can be identified with the coset $(H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)) / \text{Aut}(Q)$, see Proposition 5.5.

It is not easy in general to count elements in the coset above, but we can manage when Q is a compact surface with only one boundary. In this case, codimension-one faces sit in the boundary circle, so a characteristic function on Q is nothing but a coloring on a circle (with vertices) with three colors.

The paper is organized as follows. In section 2, we introduce the notion of locally standard 2-torus manifold and give several examples. Following Davis and Januszkiewicz [4], we define a characteristic function and construct a locally standard 2-torus manifold from a characteristic function and a principal bundle in section 3. In section 4 we describe the euler characteristic of a locally standard 2-torus manifold in terms of its orbit space. Section 5 discusses three equivalence relations among locally standard 2-torus manifolds and identify them with some cosets. We count the number of colorings on a circle in section 6. Applying this result, we find in section 7 the number of equivariant homeomorphism classes in locally standard 2-torus manifolds when the orbit space is a compact surface with only one boundary.

2. 2-TORUS MANIFOLDS

We denote the quotient additive group $\mathbb{Z}/2\mathbb{Z}$ by \mathbb{Z}_2 throughout this paper. The natural action of a 2-torus $(\mathbb{Z}_2)^n$ of rank n on \mathbb{R}^n defined by

$$(x_1, \dots, x_n) \longmapsto ((-1)^{g_1} x_1, \dots, (-1)^{g_n} x_n), \quad (g_1, \dots, g_n) \in (\mathbb{Z}_2)^n$$

is called *the standard representation* of $(\mathbb{Z}_2)^n$. The orbit space is a positive cone $\mathbb{R}_{\geq 0}^n$. Any real n -dimensional faithful representation of $(\mathbb{Z}_2)^n$ is obtained from the standard representation by composing a group automorphism of $(\mathbb{Z}_2)^n$, up to isomorphism. Therefore the orbit space of the faithful representation space can also be identified with $\mathbb{R}_{\geq 0}^n$.

A 2-torus manifold M is a closed smooth manifold of dimension n with an effective smooth action of $(\mathbb{Z}_2)^n$. We say that M is *locally standard* if for each point x in M , there is a $(\mathbb{Z}_2)^n$ -invariant neighborhood V_x of x such that V_x is equivariantly homeomorphic to an invariant open subset of a real n -dimensional faithful representation space of $(\mathbb{Z}_2)^n$.

Remark. The notion of a torus manifold is introduced in [8]. It is a closed smooth manifold of dimension $2n$ with an effective smooth action of a compact n -dimensional

torus $(S^1)^n$ having a fixed point. (More precisely speaking, an orientation data on M called an omniorientation in [2] is incorporated in the definition.) There is also a notion of local standardness in this setting ([4]). Although many interesting examples of torus manifolds are locally standard (e.g. this is the case for compact non-singular toric varieties with restricted action of the compact torus, more generally for torus manifolds with vanishing odd degree cohomology, [11]), the local standardness is not assumed in the study of [8] and [10] because a combinatorial object called a multi-fan can be defined without assuming it. However, the existence of a fixed point is not assumed for a 2-torus manifold unlike a torus manifold.

For a locally standard 2-torus manifold M , the orbit space Q of M naturally becomes a manifold with corners (see [3] for the details of a manifold with corners). Therefore the notion of a face can be defined for Q . In this paper we assume that a face is connected. We call a face of dimension 0 a *vertex*, a face of dimension one an *edge* and a codimension-one face a *facet*.

An n -dimensional convex polytope P is said to be *simple* if exactly n facets meet at each of its vertices. Each point of a simple convex polytope P has a neighborhood which is affine isomorphic to an open subset of the positive cone $\mathbb{R}_{\geq 0}^n$, so P is an n -dimensional manifold with corners. A locally standard 2-torus manifold M is said to be a *small cover* when its orbit space is a simple convex polytope, see [4].

We call a closed, connected, codimension-one submanifold of M *characteristic* if it is a connected component of the set fixed pointwise by some \mathbb{Z}_2 subgroup. Since M is compact, M has only finitely many characteristic submanifolds. The action of $(\mathbb{Z}_2)^n$ is free outside the union of all characteristic submanifolds, in other words, a point of M with non-trivial isotropy subgroup is contained in some characteristic submanifold of M .

Through the quotient map $M \rightarrow Q$, a fixed point in M corresponds to a vertex of Q and a characteristic submanifold of M corresponds to a facet of Q . A connected component of the intersection of k characteristic submanifolds of M corresponds to a codimension- k face of Q , so a codimension- k face of Q is a connected component of the intersection of k facets. In particular, any codimension-two face of Q is a connected component of the intersection of two facets of Q , which means that Q is *nice*, see [3].

We shall give examples of locally standard 2-torus manifolds.

Example 2.1. A real projective space $\mathbb{R}P^n$ with the standard $(\mathbb{Z}_2)^n$ -action defined by

$$[x_0, x_1, \dots, x_n] \longmapsto [x_0, (-1)^{g_1}x_1, \dots, (-1)^{g_n}x_n], \quad (g_1, \dots, g_n) \in (\mathbb{Z}_2)^n$$

is a locally standard 2-torus manifold. It has $n+1$ isolated points and $n+1$ characteristic submanifolds. The orbit space of $\mathbb{R}P^n$ by this action is an n -simplex, so this locally standard 2-torus manifold is actually a small cover.

Example 2.2. Let S^1 denote the unit circle in the complex plane \mathbb{C} and consider two involutions on $S^1 \times S^1$ defined by

$$t_1 : (z, w) \longmapsto (-z, w), \quad t_2 : (z, w) \longmapsto (z, \bar{w}).$$

Since t_1 and t_2 are commutative, they define a $(\mathbb{Z}_2)^2$ -action on $S^1 \times S^1$, and it is easy to see that $S^1 \times S^1$ with this action is a locally standard 2-torus manifold. It has no fixed point and the orbit space is $\mathbb{R}P^1 \times I = S^1 \times I$ where I is a closed interval.

Example 2.3. If M_1 and M_2 are both locally standard 2-torus manifolds of the same dimension, then the equivariant connected sum of them along their free orbits produces a new locally standard 2-torus manifold. For example, we take $\mathbb{R}P^2$ in Example 2.1 and $S^1 \times S^1$ in Example 2.2 and do the equivariant connected sum of them along their free orbits. The orbit space of the resulting locally standard 2-torus manifold M is the connected sum of a 2-simplex with $S^1 \times I$ at their interior points. M has five characteristic submanifolds and three of them have a fixed point but the other two have no fixed point.

If M is a locally standard 2-torus manifold of dimension n and a subgroup of $(\mathbb{Z}_2)^n$ has an isolated fixed point, then the isolated point must be fixed by the entire group $(\mathbb{Z}_2)^n$. This follows from the local standardness of M . The following is an example of a closed n -manifold with an effective $(\mathbb{Z}_2)^n$ -action which is not a locally standard 2-torus manifold.

Example 2.4. Consider two involutions on the unit sphere S^2 of $\mathbb{R} \times \mathbb{C}$ defined by

$$t_1 : (x, z) \mapsto (-x, -z), \quad t_2 : (x, z) \mapsto (x, \bar{z}).$$

Since t_1 and t_2 are commutative, they define a $(\mathbb{Z}_2)^2$ -action on S^2 . But S^2 with this action is not a locally standard 2-torus manifold because the fixed point set of $t_1 t_2$ consists of two isolated points $(0, \pm\sqrt{-1})$ but they are not fixed by the entire group $(\mathbb{Z}_2)^2$.

3. CHARACTERISTIC FUNCTIONS AND PRINCIPAL BUNDLES

Let Q be an n -dimensional nice manifold with corners. We denote by $\mathcal{F}(Q)$ the set of facets of Q . A codimension- k face of Q is a connected component of the intersection of k facets. We call a map

$$\lambda : \mathcal{F}(Q) \longrightarrow (\mathbb{Z}_2)^n$$

a *characteristic function* on Q if it satisfies the following linearly independent condition:

if a codimension- k face F of Q is a connected component of the intersection of k facets F_1, \dots, F_k , then $\lambda(F_1), \dots, \lambda(F_k)$ are linearly independent when viewed as vectors of the vector space $(\mathbb{Z}_2)^n$ over the field \mathbb{Z}_2 .

We denote by G_F the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_1), \dots, \lambda(F_k)$.

Remark. When $n \leq 2$, it is easy to see that any Q admits a characteristic function. When $n = 3$, Q admits a characteristic function if the boundary of Q is a union of 2-spheres, which follows from the Four Color Theorem, but Q may not admit a characteristic function otherwise. When $n \geq 4$, there is a simple convex polytope which admits no characteristic function, see [4, Nonexamples 1.22].

A characteristic function arises naturally from a locally standard 2-torus manifold M of dimension n with orbit space Q . A facet of Q is the image of a characteristic

submanifold of M by the quotient map $\pi: M \rightarrow Q$. To each element $F \in \mathcal{F}(Q)$ we assign the nonzero element of $(\mathbb{Z}_2)^n$ which fixes pointwise the characteristic submanifold $\pi^{-1}(F)$. The local standardness of M implies that this assignment satisfies the linearly independent condition above required for a characteristic function.

Besides the characteristic function, a principal $(\mathbb{Z}_2)^n$ -bundle over Q will be associated with M as follows. We take a small invariant open tubular neighborhood for each characteristic submanifold of M and remove their union from M . Then the $(\mathbb{Z}_2)^n$ -action on the resulting space is free and its orbit space can naturally be identified with Q , so it gives a principal $(\mathbb{Z}_2)^n$ -bundle over Q .

We have associated a characteristic function and a principal $(\mathbb{Z}_2)^n$ -bundle with a locally standard 2-torus manifold. Conversely, one can reproduce the locally standard 2-torus manifold from these two data. This is done by Davis-Januszkiewicz [4] when Q is a simple convex polytope, but their construction still works in our setting. Let $\xi = (E, \kappa, Q)$, where $\kappa: E \rightarrow Q$, be a principal $(\mathbb{Z}_2)^n$ -bundle over Q and let $\lambda: \mathcal{F}(Q) \rightarrow (\mathbb{Z}_2)^n$ be a characteristic function on Q . We define an equivalence relation \sim on E as follows: for $u_1, u_2 \in E$

$$u_1 \sim u_2 \iff \kappa(u_1) = \kappa(u_2) \text{ and } u_1 = u_2 g \text{ for some } g \in G_F$$

where F is the face of Q containing $\kappa(u_1) = \kappa(u_2)$ in its relative interior and G_F is the subgroup of $(\mathbb{Z}_2)^n$ defined at the beginning of this section. Then the quotient space E/\sim , denoted by $M(\xi, \lambda)$, naturally inherits the $(\mathbb{Z}_2)^n$ -action from E .

The following is proved in [4] when Q is a simple convex polytope, but the same proof works in our setting.

Lemma 3.1. *If a locally standard 2-torus manifold M over Q has ξ as the associated principal $(\mathbb{Z}_2)^n$ -principal bundle and λ as the characteristic function, then there is an equivariant homeomorphism from $M(\xi, \lambda)$ to M which covers the identity on Q .*

4. EULER CHARACTERISTIC OF A LOCALLY STANDARD 2-TORUS MANIFOLD

The following formula describes the euler characteristic $\chi(M)$ of a locally standard 2-torus manifold M in terms of its orbit space.

Theorem 4.1. *If M is a locally standard 2-torus manifold over Q , then*

$$\chi(M) = \sum_F 2^{\dim F} \chi(F, \partial F) = \sum_F 2^{\dim F} (\chi(F) - \chi(\partial F))$$

where F runs over all faces of Q .

Proof. As observed in Section 3, M is the disjoint union of $2^{\dim F}$ copies of $F \setminus \partial F$ over all faces F of Q . This implies the former identity in the theorem. The latter identity is well-known. In fact, it follows from the homology exact sequence for a pair $(F, \partial F)$. \square

When $\dim M = 2$, Q is a surface with boundary and each boundary component is a circle with at least two vertices if it has a vertex.

Corollary 4.2. *If $\dim M = 2$ and Q has m vertices, then $\chi(M) = 4\chi(Q) - m$.*

Proof. Since ∂Q is a union of circles, $\chi(Q, \partial Q) = \chi(Q)$. If a boundary circle has no vertex, then it is an edge without boundary and its euler characteristic is zero. So we may neglect it. If F is an edge with a vertex, then it has two endpoints and $\chi(F, \partial F) = \chi(F) - \chi(\partial F) = -1$, and if F is a vertex, then $\chi(F, \partial F) = \chi(F) = 1$. Since the number of edges with a vertex and the number of vertices are both m , it follows from Theorem 4.1 that

$$\chi(M) = 2^2 \chi(Q) - 2m + m = 4\chi(Q) - m.$$

□

Remark. When $\dim M = 2$, it is not difficult to see that M is orientable if and only if Q is orientable and the characteristic function $\lambda: \mathcal{F}(Q) \rightarrow (\mathbb{Z}_2)^2$ associated with M assigns exactly two elements to each boundary component of Q with a vertex, cf. [12]. Therefore one can find the homeomorphism type of M from the corollary above and the characteristic function λ .

5. CLASSIFICATION OF LOCALLY STANDARD 2-TORUS MANIFOLDS

In this section we introduce three notions of equivalence in locally standard 2-torus manifolds over Q and identify each set of equivalence classes with a coset of $H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)$ by some action.

Following Davis and Januszkiewicz [4] we say that two locally standard 2-torus manifolds M and M' over Q are *equivalent* if there is a homeomorphism $f: M \rightarrow M'$ together with an element $\sigma \in \mathrm{GL}(n, \mathbb{Z}_2)$ such that

- (1) $f(gx) = \sigma(g)f(x)$ for all $g \in (\mathbb{Z}_2)^n$ and $x \in M$, and
- (2) f induces the identity on the orbit space Q .

When we classify locally standard 2-torus manifolds up to the above equivalence, it suffices to consider locally standard 2-torus manifolds of the form $M(\xi, \lambda)$ by Lemma 3.1. We denote by ξ^σ the principal $(\mathbb{Z}_2)^n$ -bundle ξ with $(\mathbb{Z}_2)^n$ -action through $\sigma \in \mathrm{GL}(n, \mathbb{Z}_2)$. Then it would be obvious that $M(\xi', \lambda')$ is equivalent to $M(\xi, \lambda)$ if and only if there exists $\sigma \in \mathrm{GL}(n, \mathbb{Z}_2)$ such that $\xi' = \xi^\sigma$ and $\lambda' = \sigma \circ \lambda$.

We denote by $\mathcal{P}(Q)$ the set of all principal $(\mathbb{Z}_2)^n$ -bundles over Q . Since the classifying space of $(\mathbb{Z}_2)^n$ is an Eilenberg-MacLane space $K((\mathbb{Z}_2)^n, 1)$, $\mathcal{P}(Q)$ can naturally be identified with $H^1(Q; (\mathbb{Z}_2)^n)$ and the action of σ sending ξ to ξ^σ is nothing but the action on $H^1(Q; (\mathbb{Z}_2)^n)$ induced from the automorphism σ on the coefficient $(\mathbb{Z}_2)^n$. With this understood, the above fact implies the following.

Proposition 5.1. *The set of equivalence classes in locally standard 2-torus manifolds over Q bijectively corresponds to the coset*

$$\mathrm{GL}(n, \mathbb{Z}_2) \backslash (H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q))$$

by the diagonal action.

The action of $\mathrm{GL}(n, \mathbb{Z}_2)$ on $H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)$ is free when Q has a vertex by the following lemma.

Lemma 5.2. *If Q has a vertex, then the action of $\mathrm{GL}(n, \mathbb{Z}_2)$ on $\Lambda(Q)$ is free and $|\Lambda(Q)| = |\mathrm{GL}(n, \mathbb{Z}_2) \backslash \Lambda(Q)| \prod_{k=1}^n (2^n - 2^{k-1})$.*

Proof. Suppose that $\lambda = \sigma \circ \lambda$ for some $\lambda \in \Lambda(Q)$ and $\sigma \in \mathrm{GL}(n, \mathbb{Z}_2)$. Take a vertex of Q and let F_1, \dots, F_n be the facets of Q meeting at the vertex. Then

$$(\lambda(F_1), \dots, \lambda(F_n)) = \sigma(\lambda(F_1), \dots, \lambda(F_n)).$$

Since the matrix $(\lambda(F_1), \dots, \lambda(F_n))$ is non-singular, σ is the identity matrix. This proves the former statement in the lemma. Then the latter statement follows from the well-known fact that $|\mathrm{GL}(n, \mathbb{Z}_2)| = \prod_{k=1}^n (2^n - 2^{k-1})$, see [1]. \square

Lemma 5.2 is also helpful to count the number of elements in $\Lambda(Q)$. Here is an example.

Example 5.3. (The number of characteristic functions on a prism.) There exist seven combinatorially inequivalent 3-polytopes with six vertices (see [6, Theorem 6.7]) and only one of them is simple, which is a prism P^3 .

Let us count the number of characteristic functions on P^3 . P^3 has five facets, consisting of three square facets and two triangular facets. We denote the three square facets by F_1, F_2, F_4 , and the two triangular facets by F_3, F_5 . The facets F_1, F_2, F_3 intersect at a vertex and we may assume that a characteristic function λ on P^3 takes the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $(\mathbb{Z}_2)^3$ on F_1, F_2, F_3 respectively through the action of $\mathrm{GL}(3, \mathbb{Z}_2)$ on $(\mathbb{Z}_2)^3$. The characteristic function λ must satisfy the linearly independent condition at each vertex of P^3 . This requires that the values of λ on the remaining facets F_4, F_5 must be as follows:

$$(\lambda(F_4), \lambda(F_5)) = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3) \quad \text{or} \quad (\mathbf{e}_1 + \mathbf{e}_2, a\mathbf{e}_1 + b\mathbf{e}_2 + \mathbf{e}_3)$$

where $a, b \in \mathbb{Z}_2$. Therefore

$$|\mathrm{GL}(3, \mathbb{Z}_2) \backslash \Lambda(P^3)| = 5 \quad \text{and} \quad |\Lambda(P^3)| = 5 |\mathrm{GL}(3, \mathbb{Z}_2)| = 840$$

by Lemma 5.2.

Another natural equivalence relation among locally standard 2-torus manifolds is equivariant homeomorphism. An *automorphism* of Q is a self-homeomorphism of Q as a manifold with corners, and we denote the group of automorphisms of Q by $\mathrm{Aut}(Q)$. Similarly, an *automorphism* of $\mathcal{F}(Q)$ is a bijection from $\mathcal{F}(Q)$ to itself which preserves the poset structure of $\mathcal{F}(Q)$ defined by inclusions of faces, and we denote the group of automorphisms of $\mathcal{F}(Q)$ by $\mathrm{Aut}(\mathcal{F}(Q))$. An automorphism of Q induces an automorphism of $\mathcal{F}(Q)$, so we have a natural homomorphism

$$(5.1) \quad \Phi: \mathrm{Aut}(Q) \rightarrow \mathrm{Aut}(\mathcal{F}(Q)).$$

We note that $\mathrm{Aut}(\mathcal{F}(Q))$ acts on $\Lambda(Q)$ by sending $\lambda \in \Lambda(Q)$ to $\lambda \circ h$ for $h \in \mathrm{Aut}(\mathcal{F}(Q))$.

Lemma 5.4. *$M(\xi, \lambda)$ is equivariantly homeomorphic to $M(\xi', \lambda')$ if and only if there is an $h \in \mathrm{Aut}(Q)$ such that $\lambda' = \lambda \circ \Phi(h)$ and $h^*(\xi') = \xi$ in $\mathcal{P}(Q)$, where $h^*(\xi')$ denotes the bundle induced from ξ' by h .*

Proof. If $M(\xi, \lambda)$ is equivariantly homeomorphic to $M(\xi', \lambda')$, then there is an equivariant homeomorphism $H: M(\xi', \lambda') \rightarrow M(\xi, \lambda)$ and it is easy to see that the automorphism of Q induced from H is the desired h in the theorem.

Conversely, suppose that there is an $h \in \Lambda(Q)$ such that $\lambda' = \lambda \circ \Phi(h)$ and $\xi' = h^*(\xi)$ in $\mathcal{P}(Q)$. Then there is a bundle map $\hat{h}: \xi' \rightarrow \xi$ which covers h , and \hat{h} descends to a map H from $M(\xi', \lambda')$ to $M(\xi, \lambda)$ because $\lambda' = \lambda \circ \Phi(h)$. It is not difficult to see that H is an equivariant homeomorphism. \square

$\text{Aut}(Q)$ naturally acts on $H^1(Q; (\mathbb{Z}_2)^n)$ and the canonical bijection between $\mathcal{P}(Q)$ and $H^1(Q; (\mathbb{Z}_2)^n)$ is equivariant with respect to the actions of $\text{Aut}(Q)$.

Proposition 5.5. *The set of equivariant homeomorphism classes in all locally standard 2-torus manifolds over Q bijectively corresponds to the coset*

$$(H^1(Q, (\mathbb{Z}_2)^n) \times \Lambda(Q)) / \text{Aut}(Q)$$

by the diagonal action of $\text{Aut}(Q)$. If Q is a simple convex polytope, then the set of equivariant homeomorphism classes in all small covers over Q bijectively corresponds to the coset $\Lambda(Q) / \text{Aut}(\mathcal{F}(Q))$.

Proof. The former statement in the proposition follows from Lemma 5.4. If Q is a simple polytope, then $H^1(Q; (\mathbb{Z}_2)^n) = 0$. Therefore, the latter statement in the proposition follows if we prove that the map Φ in (5.1) is surjective when Q is a simple convex polytope.

A simple polytope Q has a simplicial polytope Q^* as its dual and the face poset $\mathcal{F}(Q)$ is same as $\mathcal{F}(Q^*)$ with reversed inclusion relation. Therefore $\text{Aut}(\mathcal{F}(Q)) = \text{Aut}(\mathcal{F}(Q^*))$. Since Q^* is simplicial, an element φ of $\text{Aut}(\mathcal{F}(Q^*))$ is realized by a simplicial automorphism on the boundary of Q^* , so it extends to an automorphism of Q^* . Since Q is dual to Q^* , the automorphism of Q^* determines a bijection on the vertex set of Q and hence an automorphism of Q which induces the chosen φ . \square

Our last equivalence relation is a combination of the previous two relations. We say that two locally standard 2-torus manifolds M and M' over Q are *weakly equivariantly homeomorphic* if there is a homeomorphism $f: M \rightarrow M'$ together with $\sigma \in \text{GL}(n, \mathbb{Z}_2)$ such that $f(gx) = \sigma(g)f(x)$ for any $g \in (\mathbb{Z}_2)^n$ and $x \in M$. We note that f induces an automorphism of Q but it may not be the identity on Q . The observation above shows that $M(\xi, \lambda)$ and $M(\xi', \lambda')$ are weakly equivariantly homeomorphic if and only if there are $h \in \text{Aut}(Q)$ and $\sigma \in \text{GL}(n, \mathbb{Z}_2)$ such that $\xi' = h^*(\xi^\sigma)$ and $\lambda' = \sigma \circ \lambda \circ h$. It follows that

Proposition 5.6. *The set of weakly equivariant homeomorphism classes in locally standard 2-torus manifolds over Q bijectively corresponds to the double coset*

$$\text{GL}(n, \mathbb{Z}_2) \backslash (H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)) / \text{Aut}(Q)$$

by the diagonal actions of $\text{Aut}(Q)$ and $\text{GL}(n, \mathbb{Z}_2)$. If Q is a simple convex polytope, then the set of weakly equivariant homeomorphism classes in small covers over Q bijectively corresponds to the double coset

$$(5.2) \quad \text{GL}(n, \mathbb{Z}_2) \backslash \Lambda(Q) / \text{Aut}(\mathcal{F}(Q)).$$

Remark. When Q is a right-angled regular hyperbolic polytope (such Q is the dodecahedron, the 120-cell or an m -gon with $m \geq 5$), it is shown in [7, Theorem 3.3] that the double coset (5.2) agrees with the set of hyperbolic structures in small covers over Q . This together with Mostow rigidity implies that when $\dim Q \geq 3$, that is, when Q is the dodecahedron or the 120-cell, the double coset (5.2) agrees with the set of homeomorphism classes in small covers over Q ([7, Corollary 3.4]), i.e., the natural surjective map from the double coset to the set of homeomorphism classes in small covers over Q is bijective for such Q . However, this last statement does not hold for an m -gon Q with $m \geq 6$ although it holds for $m = 3, 4, 5$, see the remark following Example 6.5 in the next section.

6. ENUMERATION OF COLORINGS ON A CIRCLE

When $\dim Q = 2$, each boundary component is a circle with at least two vertices if it has a vertex, and any two non-zero elements in $(\mathbb{Z}_2)^2$ form a basis of $(\mathbb{Z}_2)^2$; so a characteristic function on Q is equivalent to coloring arcs on the boundary circles with three colors in such a way that any two adjacent arcs have different colors.

Let $S(m)$ be a circle with m (≥ 2) vertices. A coloring on $S(m)$ (with three colors) means to color arcs of $S(m)$ in such a way that any adjacent arcs have different colors. We denote by $\Lambda(m)$ the set of all colorings on $S(m)$ and set

$$A(m) := |\Lambda(m)|.$$

Lemma 6.1. $A(m) = 2^m + (-1)^m 2$.

Proof. Let $L(m)$ be a segment with $m+1$ vertices including the endpoints, so $L(m)$ has m segments. The number of coloring segments of $L(m)$ with three colors in such a way that any adjacent segments have different colors is $3 \cdot 2^{m-1}$. If the two end segments have different colors, then it produces a coloring on $S(m)$ by gluing the end points of $L(m)$. If the two end segments have the same color, then it produces a coloring on $S(m-1)$ by gluing the end segments of $L(m)$. Thus, we have that

$$(6.1) \quad A(m) + A(m-1) = 3 \cdot 2^{m-1}.$$

It follows that

$$A(m) - 2A(m-1) = -(A(m-1) - 2A(m-2)) = \cdots = (-1)^{m-3}(A(3) - 2A(2))$$

and a simple observation shows that $A(3) = A(2) = 6$, so

$$(6.2) \quad A(m) - 2A(m-1) = (-1)^m 6.$$

The lemma then follows from (6.1) and (6.2). \square

We think of $S(m)$ as the unit circle of \mathbb{C} with m vertices $e^{2\pi k/m}$ ($k = 0, 1, \dots, m-1$). Let \mathfrak{D}_m be the dihedral group of order $2m$ consisting of m rotations of \mathbb{C} by angles $2\pi k/m$ ($k = 0, 1, \dots, m-1$) and m reflections with respect to lines in \mathbb{C} obtained by rotating the real axis by angles $\pi k/m$ ($k = 0, 1, \dots, m-1$). Then the action of \mathfrak{D}_m on $S(m)$ preserves the vertices so that \mathfrak{D}_m acts on the set $\Lambda(m)$. With this understood we have

Theorem 6.2. *Let φ denote the Euler's totient function, that is, $\varphi(1) = 1$ and $\varphi(N)$ for a positive integer $N (\geq 2)$ is the number of positive integers both less than N and coprime to N . Then*

$$|\Lambda(m)/\mathfrak{D}_m| = \frac{1}{2m} \left(\sum_{2 \leq d|m} \varphi(m/d) A(d) + \frac{1 + (-1)^m}{2} \cdot 3 \cdot 2^{m/2} \cdot \frac{m}{2} \right).$$

Proof. The famous Burnside Lemma or Cauchy-Frobenius Lemma (see [1]) says that if G is a finite group and X is a finite G -set, then

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where X^g denotes the set of g -fixed points in X . We apply this formula to our \mathfrak{D}_m -set $\Lambda(m)$. Let $a \in \mathfrak{D}_m$ be the rotation by angle $2\pi/m$ and $b \in \mathfrak{D}_m$ be the reflection with respect to the real axis. Then we have

$$(6.3) \quad |\Lambda(m)/\mathfrak{D}_m| = \frac{1}{2m} \sum_{k=0}^{m-1} (|\Lambda(m)^{a^k}| + |\Lambda(m)^{a^k b}|).$$

Here, if d is the greatest common divisor of k and m , then $\Lambda(m)^{a^k} = \Lambda(m)^{a^d}$ because the subgroup generated by a^k is same as that by a^d . Since $\Lambda(m)^{a^d} = \Lambda(d)$ and $\Lambda(1)$ is empty, we have

$$(6.4) \quad \sum_{k=0}^{m-1} |\Lambda(m)^{a^k}| = \sum_{2 \leq d|m} \varphi(m/d) A(d).$$

On the other hand, since $a^k b$ is a reflection with respect to the line in \mathbb{C} obtained by rotating the real axis by angle $\pi k/m$, we have

$$(6.5) \quad |\Lambda(m)^{a^k b}| = \begin{cases} 3 \cdot 2^{m/2} & \text{when } m \text{ is even and } k \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Putting (6.4) and (6.5) into (6.3), we obtain the formula in the theorem. \square

Example 6.3. As is well known, $\varphi(p^n) = p^{n-1}(p-1)$ for any prime number p and positive integer n , and $\varphi(ab) = \varphi(a)\varphi(b)$ for relatively prime positive integers a and b . We set

$$B(m) := |\Lambda(m)/\mathfrak{D}_m|.$$

Using the formula in Theorem 6.2 together with Lemma 6.1, one finds that

$$B(2) = 3, \quad B(3) = 1, \quad B(4) = 6, \quad B(5) = 3, \quad B(6) = 13,$$

$$B(7) = 9, \quad B(8) = 30, \quad B(9) = 29, \quad B(10) = 78,$$

$$B(2^k) = 2^{2^k-k-1} + 3 \cdot 2^{2^{k-1}-2} + \sum_{i=1}^k 2^{2^{i-1}-i-1}$$

$$B(p^k) = \sum_{i=1}^k \frac{1}{2p^i} (2^{p^i} - 2^{p^{i-1}})$$

$$B(2p) = \frac{1}{4p} (4^p + (3p+1)2^p + 6p - 6)$$

$$B(pq) = \frac{1}{2pq} (2^{pq} - 2^p - 2^q + 2) + \frac{1}{2p} (2^p - 2) + \frac{1}{2q} (2^q - 2)$$

where p is an odd prime number and q is another odd prime number.

Remark. The same argument as above works for coloring $S(m)$ with s colors. In this case the identity in Lemma 6.1 turns into

$$A_s(m) = (s-1)^m + (-1)^m (s-1)$$

and if we denote by $\Lambda_s(m)$ the set of all coloring on $S(m)$ with s colors, then the formula in Theorem 6.2 turns into

$$|\Lambda_s(m)/\mathfrak{D}_m| = \frac{1}{2m} \left(\sum_{2 \leq d|m} \varphi(m/d) A_s(d) + \frac{1+(-1)^m}{2} \cdot s \cdot (s-1)^{m/2} \cdot \frac{m}{2} \right).$$

The computation of $|\mathrm{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m)/\mathfrak{D}_m|$ can be done in a similar fashion to the above but is rather complicated. We note that the action of $\mathrm{GL}(2, \mathbb{Z}_2)$ on $\Lambda(m)$ is permutation of the 3 colors used to color $S(m)$. $\mathrm{GL}(2, \mathbb{Z}_2)$ consists of 6 elements and three of them are of order 2 and two of them is of order 3.

Theorem 6.4. *Let α and β be the functions defined as follows:*

$$\alpha(1) = 1, \quad \alpha(2) = 3, \quad \alpha(3) = 2, \quad \alpha(6) = 4,$$

$$\beta(1) = 0, \quad \beta(2) = 2, \quad \beta(3) = 2, \quad \beta(6) = 4.$$

Then $|\mathrm{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m)/\mathfrak{D}_m|$ is given by

$$\frac{1}{2m} \left[\sum_{d|m} \left\{ \varphi(m/d) \cdot \frac{1}{6} \left(\alpha((m/d, 6)) A(d) + \beta((m/d, 6)) A(d-1) \right) \right\} + E(m) \right]$$

where $(m/d, 6)$ denotes the greatest common divisor of m/d and 6, $A(q) = 2^q + (-1)^q 2$ as before, and

$$E(m) = \begin{cases} \frac{m}{6} A\left(\frac{m+1}{2}\right) & \text{if } m \text{ is odd,} \\ m \cdot 2^{m/2-1} & \text{if } m \text{ is even.} \end{cases}$$

Proof. Applying the Burnside Lemma to our \mathfrak{D}_m -set $\Gamma(m) := \mathrm{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m)$, we have

$$(6.6) \quad \begin{aligned} |\mathrm{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m) / \mathfrak{D}_m| &= \frac{1}{2m} \sum_{g \in \mathfrak{D}_m} |\Gamma(m)^g| \\ &= \frac{1}{2m} \sum_{k=0}^{m-1} (|\Gamma(m)^{a^k}| + |\Gamma(m)^{a^{kb}}|) = \frac{1}{2m} \left[\sum_{d|m} \varphi(m/d) |\Gamma(m)^{a^d}| + \sum_{k=0}^{m-1} |\Gamma(m)^{a^{kb}}| \right]. \end{aligned}$$

We need to analyze $|\Gamma(m)^{a^d}|$ with $d|m$ and $|\Gamma(m)^{a^{kb}}|$.

First we shall treat $|\Gamma(m)^{a^d}|$ with $d|m$. Note that $\lambda \in \Lambda(m)$ is a representative of $\Gamma(m)^{a^d}$ if and only if there is $\sigma \in \mathrm{GL}(2, \mathbb{Z}_2)$ such that

$$(6.7) \quad \sigma \circ \lambda = \lambda \circ a^d.$$

Since a^d is of order m/d , the repeated use of (6.7) shows that

$$(6.8) \quad \sigma^{m/d} = 1.$$

The identity (6.7) implies that the λ satisfying (6.7) can be determined by the coloring restricted to the union of a consecutive d arcs, say T , and it also tells us how to recover λ from the coloring on T .

Let μ be a coloring on T . We shall count colorings λ on $S(m)$ which are extensions of μ and satisfy (6.7) for some $\sigma \in \mathrm{GL}(2, \mathbb{Z}_2)$. To each σ satisfying (6.8), there is a unique extension to $S(m)$ which satisfies (6.7). However, the extended one may not be a coloring, i.e., two arcs meeting at a junction of T and its translations by rotations $(a^d)^r$ ($r = 1, \dots, m/d - 1$) may have the same color. Let t and t' be the end arcs of T such that the rotation of t by a^{d-1} is t' . (Note: When $d = 1$, we understand $t = t'$ and then the subsequent argument works.) The extended one is a coloring if and only if

$$(6.9) \quad \sigma(\mu(t)) \neq \mu(t').$$

As is easily checked, the number of σ satisfying conditions (6.8) and (6.9) is $\alpha((m/d, 6))$ if $\mu(t) \neq \mu(t')$ and is $\beta((m/d, 6))$ if $\mu(t) = \mu(t')$. On the other hand, the number of μ with $\mu(t) \neq \mu(t')$ is $A(d)$ and that with $\mu(t) = \mu(t')$ is $A(d-1)$. It follows that the number of λ satisfying (6.7) for some σ is $\alpha((m/d, 6))A(d) + \beta((m/d, 6))A(d-1)$. This proves that

$$(6.10) \quad |\Gamma(m)^{a^d}| = \frac{1}{6} \left(\alpha((m/d, 6))A(d) + \beta((m/d, 6))A(d-1) \right)$$

since the action of $\mathrm{GL}(2, \mathbb{Z}_2)$ on $\Lambda(m)$ is free by Lemma 5.2 and the order of $\mathrm{GL}(2, \mathbb{Z}_2)$ is 6.

Next we shall treat $|\Gamma(m)^{a^{kb}}|$. The argument is similar to the above. As before, $\lambda \in \Lambda(m)$ is a representative of $\Gamma(m)^{a^{kb}}$ if and only if there is $\sigma \in \mathrm{GL}(2, \mathbb{Z}_2)$ such that

$$(6.11) \quad \sigma \circ \lambda = \lambda \circ a^{kb}.$$

Since a^{kb} is of order two, the repeated use of (6.11) shows that

$$(6.12) \quad \sigma^2 = 1.$$

Suppose that m is odd. Then the line fixed by a^{kb} goes through a vertex, say v , of $S(m)$ and the midpoint of the arc, say e' , of $S(m)$ opposite to the vertex v . Let H be

the union of $(m+1)/2$ consecutive arcs starting from v and ending at e' . Let e be the other end arc of H different from e' . The arc e has v as a vertex. Let ν be a coloring on H and let $\sigma \in \text{GL}(2, \mathbb{Z}_2)$ satisfy (6.12). Then ν has an extension to a coloring of $S(m)$ satisfying (6.11) if and only if

$$\sigma(\nu(e)) \neq \nu(e) \quad \text{and} \quad \sigma(\nu(e')) = \nu(e').$$

It follows that $\nu(e)$ must be different from $\nu(e')$ and there is only one σ satisfying the two identities above for each such ν . Since the number of ν with $\nu(e) \neq \nu(e')$ is $A((m+1)/2)$, so is the number of $\lambda \in \Lambda(m)$ satisfying (6.11) for some σ . It follows that $|\Gamma(m)^{a^k b}| = \frac{1}{6}A((m+1)/2)$ and hence

$$(6.13) \quad \sum_{k=0}^{m-1} |\Gamma(m)^{a^k b}| = \frac{m}{6}A((m+1)/2).$$

Suppose that m is even and k is odd. Then the line fixed by $a^k b$ goes through the midpoints of two opposite arcs, say e and e' , of $S(m)$. Let H be the union of consecutive $m/2 + 1$ arcs starting from e and ending at e' . Let ν be a coloring on H and let $\sigma \in \text{GL}(2, \mathbb{Z}_2)$ satisfy (6.12). Then ν has an extension to a coloring of $S(m)$ satisfying (6.11) if and only if

$$\sigma(\nu(e)) = \nu(e) \quad \text{and} \quad \sigma(\nu(e')) = \nu(e').$$

If $\nu(e) \neq \nu(e')$ then such σ must be the identity, and if $\nu(e) = \nu(e')$ then there are two such σ one of which is the identity. Since the number of ν with $\nu(e) \neq \nu(e')$ is $A(m/2 + 1)$ and that with $\nu(e) = \nu(e')$ is $A(m/2)$, the number of $\lambda \in \Lambda(m)$ satisfying (6.11) for some σ is $A(m/2 + 1) + 2A(m/2)$. It follows that

$$(6.14) \quad \sum_{k=0, k:\text{odd}}^{m-1} |\Gamma(m)^{a^k b}| = \frac{m}{12}(A(m/2 + 1) + 2A(m/2)).$$

Suppose that m is even and k is even. Then the line fixed by $a^k b$ goes through two opposite vertices, say v and v' , of $S(m)$. Let H be the union of consecutive $m/2$ arcs starting from v and ending at v' . Let e and e' be the end arcs of H which respectively have v and v' as a vertex. Let ν be a coloring on H and let $\sigma \in \text{GL}(2, \mathbb{Z}_2)$ satisfy (6.12). Then ν has an extension to a coloring of $S(m)$ satisfying (6.11) if and only if

$$\sigma(\nu(e)) \neq \nu(e) \quad \text{and} \quad \sigma(\nu(e')) \neq \nu(e').$$

If $\nu(e) \neq \nu(e')$ then there is only one such σ , and if $\nu(e) = \nu(e')$ then there are two such σ . Since the number of ν with $\nu(e) \neq \nu(e')$ is $A(m/2)$ and that with $\nu(e) = \nu(e')$ is $A(m/2 - 1)$, the number of $\lambda \in \Lambda(m)$ satisfying (6.11) for some σ is $A(m/2) + 2A(m/2 - 1)$. It follows that

$$(6.15) \quad \sum_{k=0, k:\text{even}}^{m-1} |\Gamma(m)^{a^k b}| = \frac{m}{12}(A(m/2) + 2A(m/2 - 1)).$$

Thus, when m is even, it follows from (6.14) and (6.15) that

$$(6.16) \quad \sum_{k=0}^{m-1} |\Gamma(m)^{a^k b}| = \frac{m}{12} (A(m/2 + 1) + 3A(m/2) + 2A(m/2 - 1)) = m \cdot 2^{m/2-1}$$

where we used $A(q) = 2^q + (-1)^q 2$ at the latter identity.

The theorem now follows from (6.6), (6.10), (6.13) and (6.16). \square

Remark. When m is even, $\Lambda(m)$ contains exactly three colorings with two colors and it defines the unique element in the double coset $\mathrm{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m) / \mathfrak{D}_m$.

Example 6.5. We set

$$C(m) := |\mathrm{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m) / \mathfrak{D}_m|.$$

Using the formula in Theorem 6.4, one finds that

$$\begin{aligned} C(2) &= 1, & C(3) &= 1, & C(4) &= 2, & C(5) &= 1, & C(6) &= 4, & C(7) &= 3 \\ C(8) &= 8, & C(9) &= 8, & C(10) &= 18, & C(11) &= 21, & C(12) &= 48. \end{aligned}$$

We conclude this section with a remark. When Q is an m -gon ($m \geq 3$), a small cover over Q is a closed surface with euler characteristic $4 - m$ and the cardinality of the set of homeomorphism classes in small covers over Q is one (resp. two) when m is odd (resp. even). On the other hand, the double coset (5.2) agrees with $\mathrm{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m) / \mathfrak{D}_m$ and we see from Theorem 6.4 that its cardinality is strictly larger than 2 when $m \geq 6$. So, the natural surjective map from the double coset (5.2) to the set of homeomorphism classes in small covers over Q is not injective when Q is an m -gon with $m \geq 6$. However, it is bijective when $m = 3, 4, 5$, see Example 6.5.

7. LOCALLY STANDARD 2-TORUS MANIFOLDS OF DIMENSION TWO

We shall enumerate the number of equivariant homeomorphism classes in locally standard 2-torus manifolds with orbit space Q when Q is a compact surface with only one boundary.

Theorem 7.1. *Suppose that Q is a compact surface with only one boundary component with m (≥ 2) vertices and set*

$$h(Q) := |H^1(Q; (\mathbb{Z}_2)^2) / \mathrm{Aut}(Q)|.$$

Then the number of equivariant homeomorphism classes in locally standard 2-torus manifolds over Q is $h(Q)B(m)$, where $B(m) = |\Lambda(m) / \mathfrak{D}_m|$ is the number discussed in the previous section.

Proof. By Corollary 5.5 it suffices to count the number of orbits in $H^1(Q; (\mathbb{Z}_2)^2) \times \Lambda(Q)$ under the diagonal action of $\mathrm{Aut}(Q)$. Since Q has only one boundary component and m vertices, $\Lambda(Q)$ can be identified with $\Lambda(m)$ in Section 6 and $\mathrm{Aut}(\mathcal{F}(Q))$ is isomorphic to the dihedral group \mathfrak{D}_m .

Let H be the normal subgroup of $\mathrm{Aut}(Q)$ which acts on $H^1(Q; (\mathbb{Z}_2)^2)$ trivially. We claim that the restriction of the natural homomorphism

$$(7.1) \quad \mathrm{Aut}(Q) \rightarrow \mathrm{Aut}(\mathcal{F}(Q)) \cong \mathfrak{D}_m$$

to H is still surjective. An automorphism of Q (as a manifold with corners) which rotates the boundary circle and fixes the exterior of its neighborhood is an element of H . Therefore H contains all rotations in \mathfrak{D}_m . It is not difficult to see that any closed surface admits an involution which has one-dimensional fixed point component and acts trivially on the cohomology with \mathbb{Z}_2 coefficient. Since Q is obtained from a closed surface by removing an invariant open disk centered at a point in the one-dimensional fixed point set, Q admits an involution which reflects the boundary circle and lies in H . This implies the claim.

Let K be the kernel of the homomorphism $\text{Aut}(Q) \rightarrow \text{Aut}(\mathcal{F}(Q))$. Then

$$(7.2) \quad |(H^1(Q; (\mathbb{Z}_2)^2) \times \Lambda(Q)) / \text{Aut}(Q)| = |(H^1(Q; (\mathbb{Z}_2)^2) / K \times \Lambda(Q)) / \text{Aut}(Q)|.$$

For any element g in $\text{Aut}(Q)$, there is an element h in H such that gh lies in K because the map (7.1) restricted to H is surjective. Since H acts trivially on $H^1(Q; (\mathbb{Z}_2)^2)$, this shows that an $\text{Aut}(Q)$ -orbit in $H^1(Q; (\mathbb{Z}_2)^2)$ is same as an K -orbit. This means that the induced action of $\text{Aut}(Q)$ on $H^1(Q; (\mathbb{Z}_2)^2) / K$ is trivial. Therefore the right hand side at (7.2) reduces to

$$|H^1(Q; (\mathbb{Z}_2)^2) / \text{Aut}(Q)| |\Lambda(Q) / \text{Aut}(Q)|.$$

Here the first factor is $h(Q)$ by definition and the second one agrees with $|\Lambda(m) / \mathfrak{D}_m| = B(m)$ because of the surjectivity of the map (7.1), proving the theorem. \square

Example 7.2. $H^1(Q; (\mathbb{Z}_2)^2)$ is isomorphic to $H^1(Q; \mathbb{Z}_2) \oplus H^1(Q; \mathbb{Z}_2)$ and the action of $\text{Aut}(Q)$ on the direct sum is diagonal. When Q is a disk, $h(Q) = 1$. When Q is a real projective plane with an open disk removed, $H^1(Q; \mathbb{Z}_2)$ is isomorphic to \mathbb{Z}_2 and the action of $\text{Aut}(Q)$ on it is trivial. Therefore, $h(Q) = 4$ in this case. When Q is a torus with an open disk removed, $H^1(Q; \mathbb{Z}_2)$ is isomorphic to $(\mathbb{Z}_2)^2$. The action of $\text{Aut}(Q)$ on it is non-trivial and it is not difficult to see that $h(Q) = 5$ in this case.

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